Substance.	Formula.		M. P.		*m.*
Dichlorobenzol	$C_6H_4Cl_2=147$	..	52·821	..	·35933
Bromaniline	$C_6H_6BrN=172$	..	61·742	..	·35971
Trinitrotoluol......	$C_7H_5N_3O_6=227$	..	80·532	..	·35477

Here the first pair of values of *m* are almost identical. It is evident, however, that this simple relation does not generally prevail; indeed, in the case of isomeric substances, melting point may alter widely, while additive formula remains constant.

The following are examples of the identification of series by melting point:—

	M. P.		M. P.
α-Trinitrotoluol......	78·853—	α-Dinitrotoluol	69·252=9·601 }
Trinitrophenol.....	121·194—	β-Dinitrophenol	111·621=9·573 }
α-Dinitrotoluol	69·252—	Nitrotoluol	51·407=17·845 }
α-Dinitrophenol......	61·843—	α-Nitrophenol	44·392=17·451 }

The melting points recorded in the memoir are important physical constants, now first determined with a small probable error, and with an apparatus of considerable simplicity. Under no range of ordinary atmospheric pressure or latitude, and in no ordinary interval of time, are these constants likely to become impaired. Hence, if the substances referred to be prepared and preserved with average care, and handled with moderate skill, they constitute in themselves a set of thermometric standards, distributed at mean intervals of about 4° between 42° and 120°. If these substances, or most of them, be at hand, they enable an investigator to at once calibrate and directly refer to the air thermometer any standard mercurial instrument, without the necessary application of any correction whatever.

IV. "Memoir on the Theta-Functions, particularly those of Two Variables." By A. R. Forsyth, B.A., Fellow of Trinity College, Cambridge. Communicated by A. Cayley, LL.D., F.R.S. Received December 9, 1881.

(Abstract.)

The paper of which this is an abstract is divided into four parts, to the whole being prefixed a list of the more important papers dealing with the double theta-functions.

Section I treats of what may be called Rosenhain's theory, and its object is to obtain from a more general basis, and in an easier manner, the results given by Rosenhain in his essay "Mémoire sur les Fonctions des Deux Variables et à Quatre Périodes," which obtained the

prize given by the Paris Academy of Sciences in 1846, and was published in the "Mémoires des Savans Étrangers," tom. xi. Taking as the definition of the general double theta-function

$$\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}$$

$$=\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(-1)^{m\lambda+n\rho}p^{\frac{1}{4}(2m+\mu)^2}q^{\frac{1}{4}(2n+\nu)^2}r^{\frac{1}{2}(2m+\mu)(2n+\nu)}e^{(2m+\mu)\frac{i\pi x}{2K}+(2n+\nu)\frac{i\pi y}{2\Lambda}},$$

and denoting the product of four functions, in which the characteristic numbers and the variables have the subscript indices 1, 2, 3, 4 respectively, by

$$\Pi\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}$$

there is investigated, by the guidance of Professor H. J. S. Smith's paper on the Single Theta-Functions in the first volume of the "Proceedings of the London Mathematical Society," the theorem

$$4\Pi\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}=$$

$$\begin{array}{l|l}
 & +(-1)^{\Lambda'} \\
\Pi\Phi\left\{\begin{pmatrix}\Lambda, P\\ \sigma, \sigma'\end{pmatrix}X, Y\right\} & \Pi\Phi\left\{\begin{pmatrix}\Lambda, P\\ \sigma+1, \sigma'\end{pmatrix}X, Y\right\} \\
+\Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P\\ \sigma, \sigma'\end{pmatrix}X, Y\right\} & +\Pi\Phi\left\{\begin{pmatrix}\Lambda, P+1\\ \sigma+1, \sigma'\end{pmatrix}X, Y\right\} \\
+\Pi\Phi\left\{\begin{pmatrix}\Lambda, P+1\\ \sigma, \sigma'\end{pmatrix}X, Y\right\} & -\Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P\\ \sigma+1, \sigma'\end{pmatrix}X, Y\right\} \\
+\Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P+1\\ \sigma, \sigma'\end{pmatrix}X, Y\right\} & -\Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P+1\\ \sigma+1, \sigma'\end{pmatrix}X, Y\right\} \\
+(-1)^{P'} \text{ into} & +(-1)^{\Lambda'+P'} \text{ into} \\
\Pi\Phi\left\{\begin{pmatrix}\Lambda, P\\ \sigma, \sigma'+1\end{pmatrix}X, Y\right\} & \Pi\Phi\left\{\begin{pmatrix}\Lambda, P\\ \sigma+1, \sigma'+1\end{pmatrix}X, Y\right\} \\
-\Pi\Phi\left\{\begin{pmatrix}\Lambda, P+1\\ \sigma, \sigma'+1\end{pmatrix}X, Y\right\} & -\Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P\\ \sigma+1, \sigma'+1\end{pmatrix}X, Y\right\} \\
+\Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P\\ \sigma, \sigma'+1\end{pmatrix}X, Y\right\} & -\Pi\Phi\left\{\begin{pmatrix}\Lambda, P+1\\ \sigma+1, \sigma'+1\end{pmatrix}X, Y\right\} \\
-\Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P+1\\ \sigma, \sigma'+1\end{pmatrix}X, Y\right\} & +\Pi\Phi\left\{\begin{pmatrix}\Lambda+1, P+1\\ \sigma+1, \sigma'+1\end{pmatrix}X, Y\right\}
\end{array}$$

in which

$$2(\Lambda_1+\lambda_1)=2(\Lambda_2+\lambda_2)=2(\Lambda_3+\lambda_3)=2(\Lambda_4+\lambda_4)=\lambda_1+\lambda_2+\lambda_3+\lambda_4=2\Lambda',$$

and similarly for μ, ρ, ν; and

$$2(X_1+x_1)=2(X_2+x_2)=2(X_3+x_3)=2(X_4+x_4)=x_1+x_2+x_3+x_4,$$

and similarly for the y's. Since the assumption has been made that the sums of the four similarly situated numbers in the characteristics of the functions are all even, the equation comprises 4,096 cases $(=16^3)$.

This general result seems to be new, but numerous particular cases occur in Rosenhain's paper. All the important parts of his theory are deduced, viz., the quadratic relations between the constant terms in the ten even functions; the nine ratios of all but one of these by that one are expressed in terms of three independent constants k_1, k_2, k_3; and it is proved that the fifteen quotients of all the functions but one by that one can be expressed in terms of two new variables x_1, x_2, the expressions being given. The connexion between x_1, x_2, and x, y is

$$x=\int^{x_1}\frac{A+Bz}{\sqrt{Z}}\,dz+\int^{x_2}\frac{A+Bz}{\sqrt{Z}}\,dz,$$

$$y=\int^{x_1}\frac{A'+B'z}{\sqrt{Z}}\,dz+\int^{x_2}\frac{A'+B'z}{\sqrt{Z}}\,dz,$$

where $$Z=z(1-z)(1-k_1^2z)(1-k_2^2z)(1-k_3^2z),$$

and A, B, A′, B′ are perfectly determinate constants.

The quadruple periodicity is investigated at the beginning of the section; and afterwards definite-integral expressions for the periods are obtained, as, for example,

$$K=\int_0^1\frac{A+Bx}{\sqrt{X}}\,dx;$$

and it is proved that K satisfies a linear differential equation of the fourth order in each of the quantities k_1, k_2, k_3.

It may be mentioned that in dealing with the particular functions a current-number notation ϑ_0, ϑ_1, . . . , ϑ_{15} is used in preference to the cumbrous $\Phi\begin{pmatrix}0, 0\\ 0, 0\end{pmatrix}$, . . .

Section II gives the expansions of all the functions

(i) in trigonometrical series,
(ii) in ascending powers of x and y.

Much use is throughout made of a theorem

$$\Phi\left\{\begin{pmatrix}\lambda, \rho\\ \mu, \nu\end{pmatrix}x, y\right\}=e^{-\frac{2K\Lambda\log r}{\pi^2}\frac{d^2}{dx\,dy}}\,\theta_{\mu,\lambda}(x)\,\theta_{\nu,\rho}(y),$$

$(\theta_{\mu,\lambda}(x)$, $\theta_{\nu,\rho}(y)$ being single theta-functions) proved by means of the known values of the single theta-functions. From this many properties are deduced:—

(α.) The expressions for the four pairs of conjugate periods, two actual and two quasi;

(β.) The product theorem of Section I is obtained by means of the product theorem proved for single theta-functions by Professor Smith in the paper previously mentioned;

(γ.) By means of the differential equation which θ is known to satisfy (*see* "Cayley's Elliptic Functions," § 310), it is proved that the general function Φ satisfies two equations in x, y of the form

$$\frac{d^2\Phi}{dx^2}-2x\left(k'^2-\frac{\mathrm{E}}{\mathrm{K}}\right)\frac{d\Phi}{dx}+2kk'^2\frac{d\Phi}{dk}=0,$$

k, k', E having the usual connexion with $\theta_{\mu,\,\lambda}(x)$. These equations are also investigated from the definition as well as

$$r\frac{d\Phi}{dr}+\frac{2\mathrm{K}\Lambda}{\pi^2}\,\frac{d^2\Phi}{dx\,dy}=0,$$

which it is obvious from the theorem of this section that Φ satisfies.

(δ.) Expressions for all the constants occurring in the expansions of all the functions in powers of x, y are obtained. If we write

$$\vartheta_0=c_0-\frac{1}{2\,!}(\mathrm{B}_{0,0},\,\mathrm{B}_{0,1},\,\mathrm{B}_{0,2})(x,\,y)^2+\;\ldots$$
$$+\frac{(-1)^n}{2n\,!}(\mathrm{N}_{0,0},\,\mathrm{N}_{0,1},\,\ldots,\,\mathrm{N}_{0,s},\,\ldots,\,\mathrm{N}_{0.2n})(x,\,y)^{2n}+\;\ldots,$$

it is proved that

$$c_0=\Delta_1\,.\,\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}},$$

$$\mathrm{N}_{0,2s}=\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2s}\frac{d^n}{dp'^{n-s}dq'^{s}}c_0,$$

$$\mathrm{N}_{0,2s+1}=\frac{r'\mathrm{K}\Lambda}{\pi^2}\left(\frac{\pi}{\mathrm{K}}\right)^{2(n-s)}\left(\frac{\pi}{\Lambda}\right)^{2(s+1)}\frac{d^{n+1}}{dp'^{n-s}dq'^{s+1}}\Delta_2\,.\,\mathrm{K}^{\frac{1}{2}}\Lambda^{\frac{1}{2}},$$

where $\qquad p'=\log p,\qquad q'=\log q,\qquad r'=2\log r,$

$$\Delta_1=\frac{2}{\pi}\cosh\left\{r'\left(\frac{d^2}{dp'dq'}\right)^{\frac{1}{2}}\right\},$$

with a similar expression for Δ_2.

Section III forms the expression of the addition-theorem. Although no addition-theorem proper exists for theta-functions, that is to say, although $\Phi\,(x+\xi,\,y+\eta)$ cannot be written down in terms of functions of x, y and of ξ, η, an expression is obtainable in every case for

$$\Phi(x+\xi,\,y+\eta)\Phi'(x-\xi,\,y-\eta),$$

Φ, Φ' being either the same or different functions. Since any one

function of the sum of two pairs of variables may be combined in a product with any one function of the difference of the same pairs, 256 equations are necessary to give the complete expression of the theorem. These are written down in sixteen sets of sixteen each, that which is common to each set being the function of the difference of the pairs of variables. Denoting by

$$\Theta \ldots \vartheta(x+\xi, y+\eta),$$

$$\Theta' \ldots \vartheta(x-\xi, y-\eta),$$

$$\vartheta \ldots \vartheta(x, y),$$

$$\theta \ldots \vartheta(\xi, \eta),$$

one such equation is

$$c_0^2\Theta_0\Theta_0' = \theta_0^2\vartheta_0^2 + \theta_7^2\vartheta_7^2 + \theta_{10}^2\vartheta_{10}^2 + \theta_{13}^2\vartheta_{13}^2$$

where c_0 is the value of ϑ_0 when x, y are both zero. The obvious analogy with the case of the single theta-functions

$$\Theta^2(0)\Theta(u+v)\Theta(u-v) = \Theta^2(u)\Theta^2(v) - \mathrm{H}^2(u)\mathrm{H}^2(v)$$

(using the ordinary notation) need hardly be pointed out.

In Section IV many of the properties already proved for the double theta-functions are generalised for the "r" tuple theta-functions. Among these are:—

(α.) The periodicity;

(β.) The product theorem, which gives the product of four functions as the sum of 4^r products of four functions; and from it several general equations are deduced;

(γ.) The analogue of the main theorem of Section II, which is for the "r" tuple functions

$$\Phi\begin{Bmatrix}\lambda_1, \lambda_2, \ldots, \lambda_r \\ \nu_1, \nu_2, \ldots, \nu_r\end{Bmatrix} x_1, x_2, \ldots, x_r \Big\} = e^{-\frac{2}{\pi^2}\sum\limits_{s=1}^{s=r}\sum\limits_{s=1}^{t=r} \mathrm{K}_s\mathrm{K}_t \log p_{s,t} \frac{d^2}{dx_s dx_t}} \prod_{t=1}^{t=r} \theta_{\nu_t, \lambda_t}(x_t),$$

and this is used, as before, to obtain

(δ.) The r differential equations of the form

$$\frac{d^2\Phi}{dx_r} - 2x_r\left(k_r'^2 - \frac{\mathrm{E}_r}{\mathrm{K}_r}\right)\frac{d\Phi}{dx_r} + 2k_r k_r'^2 \frac{d\Phi}{dk_r} = 0,$$

and the $\frac{1}{2}r(r-1)$ of the form

$$p_{s,t}\frac{d\Phi}{dp_{s,t}} + \frac{2\mathrm{K}_s\mathrm{K}_t}{\pi^2}\frac{d^2\Phi}{dx_s dx_t} = 0,$$

all satisfied by Φ; and to indicate a method of obtaining the constants in the expansions of the Φ's in powers of the x's.